Transpose of a Matrix & Symmetric Matrix

1 Mark Questions

1. Write 2 × 2 matrix which is both symmetric and skew-symmetric matrices. **Delhi 2014C**

A null matrix of order 2 × 2 is both symmetric and skew-symmetric matrices.

For a symmetric matrix,

$$a_{ij} = a_{ji} \quad \ldots(i)$$

and for a skew-symmetric matrix,

$$a_{ij} = -a_{ji} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we get $a_{ij} = 0$ **(1)**

2. For what value of x, is the matrix

$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix?

All India 2013: HOTS

> If A is a skew-symmetric matrix, then $A = -A^T$, where A^T is transpose of matrix A.

Given, $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$

We know that, if A is a skew-symmetric matrix, then

$$A = -A^T \quad \ldots(i)$$

From Eq. (i), we get

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \quad \textbf{(1/2)}$$

On comparing the corresponding element, we get

$$x = 2 \quad \textbf{(1/2)}$$

3. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^T - B^T$. **All India 2012**

Given, $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Transpose of $B = B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$ **(1/2)**

Now, $A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3+1 & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$ **(1/2)**

4. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then find $A + A'$. **All India 2010C**

Firstly, we find the transpose of matrix A and then add the corresponding elements of both matrices A and A'.

Given, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\therefore$ $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Now, $A + A' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$ **(1)**

5. If $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$, then find $A + A'$, where A' is transpose of A. **All India 2009C**

Do same as Que 4. $\left[\textbf{Ans.} \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix}\right]$

6. If matrix $A = [1 \quad 2 \quad 3]$, then write AA'.

Delhi 2009; HOTS

> Firstly, we write the transpose of matrix A of order 1×3, whose order is 3×1, then multiply if matrix multiplication is possible to get required answer.

Given, matrix is $A = [1 \quad 2 \quad 3]$

$$\therefore \quad A' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Now,

$$AA' = [1 \quad 2 \quad 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= [(1 \times 1) + (2 \times 2) + (3 \times 3)]$$

$$= [1 + 4 + 9] = [14] \qquad \textbf{(1)}$$

4 Mark Questions

7. For the following matrices A and B, verify that

$$[AB]' = B'A'; \quad A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = [-1 \quad 2 \quad 1]$$

All India 2010

Given, $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = [-1 \quad 2 \quad 1]$

To verify $(AB)' = B'A'$

Here, $$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}_{3 \times 1} [-1 \quad 2 \quad 1]_{1 \times 3}$$

$$\Rightarrow \quad AB = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

[multiplying row by column]

$$\therefore \quad \text{LHS} = (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad \ldots \text{(i)}$$

[interchanging rows and columns] **(1½)**

Now, $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ and $A' = [1 \quad -4 \quad 3]$ **(1)**

$$\therefore \quad \text{RHS} = B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [1 \quad -4 \quad 3]$$

$$= \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad \ldots \text{(ii)}$$

[multiplying row by column]**(1)**

From Eqs. (i) and (ii), we get

$$(AB)' = B'A'$$

$\therefore$ LHS = RHS **(1/2)**

8. Express the following matrix as a sum of a symmetric and a skew-symmetric matrices and verify your result.

$$\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

All India 2010; HOTS

> Write the given matrix A as $A = P + Q$, where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$. Also, verify that P is a symmetric matrix and Q is a skew-symmetric matrix.

Let $A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Let us introduce two matrices P and Q, such that

$$P = \frac{1}{2}(A + A') \text{ and } Q = \frac{1}{2}(A - A')$$

We will show that $A = (P + Q)$

Now, $P = \frac{1}{2}(A + A')$

$$\Rightarrow P = \frac{1}{2}\left(\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} \qquad \textbf{(1)}$$

and $\quad P' = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = P$

$\therefore \quad P' = P$

So, P is a symmetric matrix. **(1)**

Now, $Q = \frac{1}{2}(A - A')$

$$\left(\begin{bmatrix} 3 & -2 & -4 \end{bmatrix} \quad \begin{bmatrix} 3 & 3 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

and $$Q' = \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= -Q$$

$\Rightarrow \quad Q' = -Q$

$\therefore$ Q is a skew-symmetric matrix. **(1)**

Now,

$$P + Q = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -4 & -8 \\ 6 & -4 & -10 \\ -2 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Thus, $\quad P + Q = A$ **(1)**